

GRUENHAGE COMPACTA AND STRICTLY CONVEX DUAL NORMS

RICHARD J. SMITH

ABSTRACT. We prove that if K is a Gruenhage compact space then $\mathcal{C}(K)^*$ admits an equivalent, strictly convex dual norm. As a corollary, we show that if X is a Banach space and $X^* = \overline{\text{span}}^{\|\cdot\|}(K)$, where K is a Gruenhage compact in the w^* -topology and $\|\cdot\|$ is equivalent to a coarser, w^* -lower semicontinuous norm on X^* , then X^* admits an equivalent, strictly convex dual norm. We give a partial converse to the first result by showing that if Υ is a tree, then $\mathcal{C}_0(\Upsilon)^*$ admits an equivalent, strictly convex dual norm if and only if Υ is a Gruenhage space. Finally, we present some stability properties satisfied by Gruenhage spaces; in particular, Gruenhage spaces are stable under perfect images.

1. INTRODUCTION AND PRELIMINARIES

In renorming theory, we determine the extent to which the norm of a given Banach space can be modified, in order to improve the geometry of the corresponding unit ball. Naturally, the structural theory of Banach spaces plays an important part in this field but, in recent times, there has been a move toward a more non-linear, topological approach. This new outlook led to the solution of some long-standing problems, as well as producing some completely unexpected results.

Recall that a norm $\|\cdot\|$ on a real Banach space X is called *strictly convex*, or *rotund*, if $\|x\| = \|y\| = \frac{1}{2}\|x+y\|$ implies $x = y$. We say that $\|\cdot\|$ is *locally uniformly rotund*, or *LUR*, if, given a point x and a sequence (x_n) in the unit sphere S_X satisfying $\|x + x_n\| \rightarrow 2$, we have $x_n \rightarrow x$ in norm. If $\|\cdot\|$ is a dual norm on X^* then $\|\cdot\|$ is called *w^* -LUR* if, given x and (x_n) as above, we have $x_n \rightarrow x$ in the w^* -topology. For a dual norm, evidently $\text{LUR} \Rightarrow w^*\text{-LUR} \Rightarrow \text{strictly convex}$.

It turns out that, in some contexts, these ostensibly convex, geometrical properties of the norm can be characterised relatively simply in purely non-linear, topological terms. Given a compact, Hausdorff space K , we denote the Banach space of continuous real-valued functions on K by $\mathcal{C}(K)$, and identify $\mathcal{C}(K)^*$ with the space of regular, signed Borel measures on K . Raja proved that if K is a compact space then $\mathcal{C}(K)^*$ admits an equivalent, dual LUR norm if and only if K is σ -discrete [8]; that is, K is a countable union of sets, each of which is discrete in its subspace topology. Moreover, $\mathcal{C}(K)^*$ admits an equivalent w^* -LUR norm if and

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only if K is descriptive [9]; the definition of a descriptive compact space is given below. Raja also proved that X^* admits an equivalent w^* -LUR norm if and only if B_{X^*} is descriptive in the w^* -topology.

Regarding strictly convex norms, the authors of [6] recently showed that X , which can be a dual space, admits an equivalent, strictly convex, $\sigma(X, N)$ -lower semicontinuous norm if and only if the square B_X^2 has a certain linear, topological decomposition with respect to a given norming subspace $N \subseteq X^*$. In this paper, we examine what can be done without the linearity, and without explicit reference to the square. Using Gruenhage compacta, we obtain a sufficient condition for a dual space X^* to admit an equivalent, strictly convex dual norm. This condition covers all established classes of Banach space known to be so renormable, including the duals of all *weakly countably determined*, or Vařák, spaces. It also covers the more general class of ‘descriptively generated’ dual spaces, introduced recently in [7].

We define descriptive compact spaces and related notions. All topological spaces are assumed to be Hausdorff. A family of subsets \mathcal{H} of a topological space X is called *isolated* if, given $H \in \mathcal{H}$, there exists an open set U that includes H and misses every other element of \mathcal{H} ; i.e. \mathcal{H} is discrete in the union $\bigcup \mathcal{H}$. The family \mathcal{H} is called a *network* for K if, given $t \in U$, where U is open, there exists $H \in \mathcal{H}$ such that $t \in H \subseteq U$. In other words, a network is a basis, but without the requirement that its elements be open subsets. Finally, we say that a compact space K is *descriptive* if it has a network \mathcal{H} that is σ -isolated; that is, $\mathcal{H} = \bigcup_n \mathcal{H}_n$, where each \mathcal{H}_n is a isolated family.

The class of descriptive compact spaces is rather large. It includes two classes of topological spaces that have featured prominently in non-separable Banach space theory, namely *Eberlein* and *Gul’ko* compacta; see, for example [2]. It also includes all σ -discrete compact spaces; in particular, all compact K such that the Cantor derivative $K^{(\omega_1)}$ is empty, where ω_1 is the least uncountable ordinal. More information about descriptive compact spaces can be found in [7].

More generally, we say that a topological space X is *fragmentable* if there exists a metric d on K , with the property that given $\varepsilon > 0$ and non-empty $E \subseteq T$, there is an open set U such that $U \cap E$ is non-empty and the d -diameter of $E \cap U$ does not exceed ε . General fragmentable compact spaces are not particularly well-behaved from the point of view of renorming. Indeed, since every scattered space is fragmented by the discrete metric, the compact $\omega_1 + 1$ is fragmentable, and it is well-known that $\mathcal{C}(\omega_1 + 1)^*$ does not admit an equivalent, strictly convex dual norm; see, for example [1, Theorem VII.5.2]. On the other hand, if X^* does admit an equivalent, strictly convex dual norm, then B_{X^*} is fragmentable in the w^* -topology [11].

The class of Gruenhage compact spaces fits between those of descriptive and fragmentable spaces.

Definition 1.1 (Gruenhage [4]). A topological space X is called a *Gruenhage* space if there exist families $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of open sets such that given distinct $x, y \in X$, there exists $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$ with two properties:

- (1) $U \cap \{x, y\}$ is a singleton;
- (2) either x lies in finitely many $U' \in \mathcal{U}_n$ or y lies in finitely many $U' \in \mathcal{U}_n$.

If we were to follow Gruenhage’s definition to the letter, the sequence (\mathcal{U}_n) above would have to cover X as well, but this demand is not necessary as property (1)

forces the sequence to cover all points of X , with at most one exception. Gruenhage calls such sequences σ -*distributively point-finite* T_0 -*separating covers* of X .

In the next section, we investigate the role of Gruenhage spaces in renorming theory. In the third section, we give a partial converse to Theorem 2.6, the principal result of the second section, and, by virtue of examples, get some measure of the gap between descriptive compact spaces and Gruenhage compact spaces. The last section is devoted to proving certain stability properties of the class of Gruenhage spaces and its subclass of compact spaces.

2. GRUENHAGE COMPACTA AND RENORMING

We shall say that a family \mathcal{H} of subsets of a topological space X *separates points* if, given distinct $x, y \in X$, there exists $H \in \mathcal{H}$ such that $\{x, y\} \cap H$ is a singleton. It should be noted that some authors demand more of point separation, namely that H can be chosen in such a way that $\{x, y\} \cap H = \{x\}$.

The next proposition brings together some equivalent formulations of Gruenhage's definition that will be of use to us.

Proposition 2.1. *Let X be a topological space. The following are equivalent.*

- (1) X is a Gruenhage space;
- (2) there exists a sequence (A_n) of closed sets and a sequence (\mathcal{H}_n) of families such that $\bigcup_n \mathcal{H}_n$ separates points, and furthermore each element of \mathcal{H}_n is an open subset of A_n and disjoint from every other element of \mathcal{H}_n ;
- (3) there exists a sequence (\mathcal{U}_n) of families of open subsets of X and sets R_n , such that $\bigcup_n \mathcal{U}_n$ separates points and $U \cap V = R_n$ whenever $U, V \in \mathcal{U}_n$ are distinct.

Proof. (1) \Rightarrow (2) follows directly from [15, Proposition 7.4]. Suppose that (2) holds. To obtain (3), simply define $R_n = K \setminus A_n$ and set $\mathcal{U}_n = \{H \cup R_n \mid H \in \mathcal{H}_n\}$. Finally, if (3) holds, define $\mathcal{V}_n = \{R_n\}$. Given distinct $x, y \in X$, there exists n and $U \in \mathcal{U}_n$ such that $\{x, y\} \cap U$ is a singleton. Let us assume that $x \in U$. There are two cases. If $x \in R_n$ then $y \notin R_n$ because $R_n \subseteq U$, thus $\{x, y\} \cap R_n$ is a singleton and, since \mathcal{V}_n is a singleton, x is in exactly one element of \mathcal{V}_n . Alternatively, we assume that $x \notin R_n$. Then $x \in V \in \mathcal{U}_n$ forces $V = U$. Hence x is in exactly one element of \mathcal{U}_n . This shows that X is Gruenhage. \square

The second formulation presented in the proposition above prompts the following definition.

Definition 2.2. Let X be a topological space. We call (A_n, \mathcal{H}_n) a *legitimate system* if A_n and (\mathcal{H}_n) are as in Proposition 2.1, part (2). We say that $\mathcal{H} = \bigcup_n \mathcal{H}_n$ is the *union* of the system.

The next result follows easily.

Corollary 2.3. *A descriptive compact space is Gruenhage.*

Proof. In [9], Raja shows that if K is a descriptive compact space then there exists a legitimate system (A_n, \mathcal{H}_n) such that its union \mathcal{H} is a network for K . \square

We will spend a little time preparing our legitimate systems for battle. We can and do assume for the rest of this section that every legitimate system (A_n, \mathcal{H}_n) , with union \mathcal{H} , satisfies three properties:

- (1) \mathcal{H} is closed under the taking of finite intersections;
- (2) $K \setminus A_n \in \mathcal{H}$ for all n ;
- (3) $A_n \setminus \bigcup \mathcal{H}_n \in \mathcal{H}$ for all n .

Indeed, we first extend the system (A_n, \mathcal{H}_n) by adding the pairs $(K, \{K \setminus A_n\})$ and $(A_n \setminus \bigcup \mathcal{H}_n, \{A_n \setminus \bigcup \mathcal{H}_n\})$ for every n . We denote the extended system again by (A_n, \mathcal{H}_n) and then consider, for each non-empty, finite $F \subseteq \mathbb{N}$, the pairs (A_F, \mathcal{H}_F) , where $A_F = \bigcap_{n \in F} A_n$ and

$$\mathcal{H}_F = \{\bigcap_{n \in F} H_n \mid H_n \in \mathcal{H}_F\}.$$

A family \mathcal{H} of pairwise disjoint subsets of K is called *scattered* if there exists a well-ordering $(H_\xi)_{\xi < \lambda}$ of \mathcal{H} such that $\bigcup_{\xi < \alpha} H_\xi$ is open in $\bigcup \mathcal{H}$ for all $\alpha < \lambda$. Equivalently, \mathcal{H} is scattered if, given non-empty $M \subseteq \bigcup \mathcal{H}$, there exists $H \in \mathcal{H}$ such that $M \cap H$ is non-empty and open in M . Scattered families naturally generalise isolated ones. The following lemma is a simple extension of Rudin's result that Radon measures on scattered compact spaces are atomic. We can state it in greater generality than required, without compromising the simplicity of the proof. We will say that $H \subseteq K$ is *universally Radon measurable* (uRm) if, given positive $\mu \in \mathcal{C}(K)^*$, there exist Borel sets E, F such that $E \subseteq H \subseteq F$ and $\mu(E) = \mu(F)$; equivalently, H can be measured by the completion of each such μ , which we again denote by μ .

Lemma 2.4. *If \mathcal{H} is a scattered family of uRm subsets of a compact space K then $\bigcup \mathcal{H}$ is uRm and $\mu(\bigcup \mathcal{H}) = \sum_{H \in \mathcal{H}} \mu(H)$ for every positive $\mu \in \mathcal{C}(K)^*$.*

Proof. Take a well-ordering $(H_\xi)_{\xi < \lambda}$ of \mathcal{H} and open sets U_α , $\alpha < \lambda$, such that $H_\alpha \subseteq U_\alpha$ and $U_\alpha \cap H_\beta$ is empty whenever $\alpha < \beta$. We proceed by transfinite induction on λ ; note that by σ -additivity, we can assume that λ is a limit ordinal of uncountable cofinality. Set $D_\alpha = U_\alpha \setminus \bigcup_{\xi < \alpha} U_\xi$ for $\alpha < \lambda$. Given positive $\mu \in \mathcal{C}(K)^*$, by the uncountable cofinality, let $\alpha < \lambda$ such that $\mu(D_\beta) = 0$ for $\alpha \leq \beta < \lambda$. The regularity of μ ensures that $\mu(D) = 0$, where $D = \bigcup_{\alpha \leq \beta < \lambda} D_\beta$. By inductive hypothesis, there exist Borel sets E, F such that $E \subseteq \bigcup_{\xi < \alpha} H_\xi \subseteq F$ and $\mu(E) = \mu(F) = \sum_{\xi < \alpha} \mu(H_\xi)$, so the conclusion follows when we consider E and $F \cup D$. \square

It is evident that, given a legitimate system (A_n, \mathcal{H}_n) , the family $\mathcal{H}'_n = \mathcal{H}_n \cup \{K \setminus A_n, A_n \setminus \bigcup \mathcal{H}_n\}$ is scattered and has union K . Moreover, the family

$$\mathcal{D}_n = \{\bigcap_{i \leq n} H_i \cap \dots \cap H_n \mid H_i \in \mathcal{H}'_i\}$$

also enjoys these properties. Readers familiar with related literature will recognise that these families lead directly to fragmentability, via Ribarska's characterisation of fragmentable spaces [10]. Elements of the proof of the following result appear in [15]. We denote both canonical norms on $\mathcal{C}(K)$ and $\mathcal{C}(K)^{**}$ by $\|\cdot\|_\infty$, and that of $\mathcal{C}(K)^*$ by $\|\cdot\|_1$. We will be identifying certain subsets of K with their indicator functions, either in $\mathcal{C}(K)$ or $\mathcal{C}(K)^{**}$.

Lemma 2.5. *Let (A_n, \mathcal{H}_n) be a legitimate system that separates points, with union \mathcal{H} and which satisfies properties (1) – (3) above. Then $N = \overline{\text{span}}^{\|\cdot\|_\infty}(\mathcal{H})$ is a subalgebra of $\mathcal{C}(K)^{**}$ that is 1-norming for $\mathcal{C}(K)^*$.*

Proof. Let \mathcal{D}_n be the families introduced above, with union \mathcal{D} . As \mathcal{H} separates points, so does \mathcal{D} . If $\mu \in \mathcal{C}(K)^*$ has variation $|\mu|$ then we have $\|\mu\|_1 =$

$\sum_{D \in \mathcal{D}_n} |\mu|(D)$ by Lemma 2.4. Thus, given $\varepsilon > 0$, we can take finite subsets $\mathcal{F}_n \subseteq \mathcal{D}_n$ and compact subsets $K_D \subseteq D$, $D \in \mathcal{F}_n$, such that $\sum_n |\mu|(K \setminus \bigcup_{D \in \mathcal{F}_n} K_D) < \varepsilon$. Put $M = \bigcap_n \bigcup_{D \in \mathcal{F}_n} K_D$ and $M_D = M \cap K_D = M \cap D$. If $\mathcal{M}_n = \{M_D \mid D \in \mathcal{F}_n\}$ then \mathcal{M}_n is family of pairwise disjoint sets with union M , and \mathcal{M}_{n+1} refines \mathcal{M}_n . Moreover, each M_D is clopen in M and, as \mathcal{D} separates points of K , so $\mathcal{M} = \bigcup_n \mathcal{M}_n$ separates points of M . Therefore, by the Stone-Weierstrass Theorem, $\mathcal{C}(M) = \overline{\text{span}}^{\|\cdot\|_\infty}(\mathcal{M})$.

It follows that we can take non-empty, disjoint $M_{D_i} \in \mathcal{M}$ and $a_i \in [1, -1]$, $i \leq n$, such that $|\mu|(M) - \sum_{i \leq n} a_i \mu(M_{D_i}) < \varepsilon$. Now $M_{D_i}, M_{D_j} \neq \emptyset$ and $M_{D_i} \cap M_{D_j} = \emptyset$ implies $D_i \cap D_j = \emptyset$. Therefore, $\sum_{i \leq n} |\mu|(D_i \setminus M_{D_i}) \leq |\mu|(K \setminus M) < \varepsilon$. We conclude that $\|\mu\|_1 - \sum_{i \leq n} a_i \mu(D_i) < 2\varepsilon$. Since $\mathcal{D} \subseteq \mathcal{H}$, we are done. \square

We say that a norm $\|\cdot\|$ on X is *pointwise uniformly rotund*, or *p-UR*, if there exists a separating subspace $F \subseteq X^*$ such that, given sequences (x_n) and (y_n) satisfying $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \rightarrow 2$, then $f(x_n - y_n) \rightarrow 0$ for all $f \in F$; see, for example [12]. Evidently, p-UR norms are strictly convex. We can now present the main theorem.

Theorem 2.6. *If K is a Gruenhage compact then:*

- (1) $\mathcal{C}(K)^*$ admits an equivalent, strictly convex, dual lattice norm;
- (2) $\mathcal{C}(K)^*$ admits an equivalent, dual p-UR norm.

Proof. The lattice norm is constructed first. We take a legitimate system (A_n, \mathcal{H}_n) satisfying the conclusion of Lemma 2.5. For $\mu \in \mathcal{C}(K)^*$ and $m \geq 1$, define the seminorm

$$\|\mu\|_{n,m}^2 = \inf \{ m^{-1} \sum_{H \in \mathcal{H}_n} |\lambda|(H)^2 + \|\mu - \lambda\|_1^2 \mid \lambda \in \mathcal{C}(A_n)^* \}.$$

We observe that $\|\mu\|_{n,m} \leq \|\mu\|_1$ and that $\|\cdot\|_{n,m}$ is w^* -lower semicontinuous. We can verify the lower semicontinuity by applying a compactness argument. Alternatively, if we denote the open set $(\bigcup \mathcal{H}_n) \cup (K \setminus A_n)$ by U , we observe that $\|\mu\|_{n,m} = \sup \{ \mu(f) \mid f \in B \}$, where

$$B = \{ f \in \mathcal{C}_0(U) \mid m \sum_{H \in \mathcal{H}_n} \|f|_H\|_\infty^2 + \|f\|_\infty^2 \leq 1 \}.$$

In this way, we see that $\|\cdot\|_{n,m}$ is also a lattice seminorm.

We define a dual lattice norm on $\mathcal{C}(K)^*$ by setting

$$\|\mu\|^2 = \|\mu\|_1^2 + \sum_{n,m} 2^{-n-m} \|\mu\|_{n,m}^2.$$

Now suppose that $\|\mu\| = \|\nu\| = \frac{1}{2} \|\mu + \nu\|$. A standard convexity argument (cf. [1, Fact II.2.3]) yields

$$(1) \quad 2\|\mu\|_{n,m}^2 + 2\|\nu\|_{n,m}^2 - \|\mu + \nu\|_{n,m}^2 = 0$$

for all n and m . By appealing to compactness or the Hahn-Banach Theorem, there exist $\mu_{n,m}, \nu_{n,m} \in \mathcal{C}(A_n)^*$ such that

$$\|\mu\|_{n,m}^2 = m^{-1} \sum_{H \in \mathcal{H}_n} |\mu_{n,m}|(H)^2 + \|\mu - \mu_{n,m}\|_1^2$$

and likewise for ν . Hence, by applying further standard convexity arguments to equation (1), we obtain

$$(2) \quad 2|\mu_{n,m}|(H)^2 + 2|\nu_{n,m}|(H)^2 - |\mu_{n,m} + \nu_{n,m}|(H)^2 = 0$$

for all n, m and $H \in \mathcal{H}_n$. Now we estimate

$$\begin{aligned}
\|\mu|_{A_n} - \mu_{n,m}\|_1 &= \|\mu - \mu_{n,m}\|_1 - \|\mu|_{K \setminus A_n}\|_1 \\
&\leq \|\mu\|_{n,m} - \|\mu|_{K \setminus A_n}\|_1 \\
&\leq [m^{-1} \sum_{H \in \mathcal{H}_n} |\mu|(H)^2 + \|\mu|_{K \setminus A_n}\|_1^2]^{\frac{1}{2}} - \|\mu|_{K \setminus A_n}\|_1 \\
&\leq m^{-\frac{1}{2}}
\end{aligned}$$

because $\|\cdot\|_1 \leq \|\cdot\|$. A similar result holds for ν . Therefore, we conclude from equation (2) that

$$2|\mu|(H)^2 + 2|\nu|(H)^2 - |\mu + \nu|(H)^2 = 0$$

for all $H \in \mathcal{H}_n$ and $n \in \mathbb{N}$. As N from Lemma 2.5 is norming, we certainly obtain $|\mu| = |\nu| = \frac{1}{2}|\mu + \nu|$. This gives $\mu = \nu$ by the following lattice argument, included for completeness. If $\lambda = \mu_+ - \nu_-$ then $|\mu| = |\nu|$ implies $\lambda = \nu_+ - \mu_-$, meaning $\mu + \nu = 2\lambda$. Hence $\mu_+ + \mu_- = |\mu| = \frac{1}{2}|\mu + \nu| = \lambda_+ + \lambda_-$. We see that $\lambda_+ = (\mu_+ - \nu_-)_+ \leq (\mu_+)_+ = \mu_+$, hence $\mu_+ = \lambda_+$ and $\mu_- = \lambda_-$. We conclude that $\mu = \nu$ as claimed.

Now we construct the p-UR norm, using the norming subspace N . First, we claim that $\|\cdot\|$ above already satisfies the p-UR property if μ_k and ν_k are positive. Suppose that μ_k and ν_k are positive measures such that $\|\mu_k\| = \|\nu_k\| = 1$ and $\|\mu_k + \nu_k\| \rightarrow 2$. As above, we can find $\mu_{k,n,m}, \nu_{k,n,m} \in \mathcal{C}(A_n)^*$ such that

$$\|\mu_k\|_{k,n,m}^2 = m^{-1} \sum_{H \in \mathcal{H}_n} |\mu_{k,n,m}|(H)^2 + \|\mu_k - \mu_{k,n,m}\|_1^2$$

and likewise for ν_k . By convexity arguments, we obtain

$$(3) \quad 2|\mu_{k,n,m}|(H)^2 + 2|\nu_{k,n,m}|(H)^2 - |\mu_{k,n,m} + \nu_{k,n,m}|(H)^2 \rightarrow 0$$

as $k \rightarrow \infty$. Moreover, if $H \in \mathcal{H}_n$, we estimate

$$|\mu_k - \mu_{k,n,m}|(H) \leq \|\mu_k|_{A_n} - \mu_{k,n,m}\|_1 \leq m^{-\frac{1}{2}}$$

and likewise for ν_k . Therefore, by fixing m large enough and appealing to equation (3), we get

$$2\mu_k(H)^2 + 2\nu_k(H)^2 - (\mu_k + \nu_k)(H)^2 \rightarrow 0$$

whence $(\mu_k - \nu_k)(H) \rightarrow 0$. It follows that $\xi(\mu_k - \nu_k) \rightarrow 0$ for all $\xi \in N$, thus completing the claim.

Now we set $\|\mu\| = \|\mu_+\| + \|\mu_-\|$. To see that this defines a dual norm, observe that as $\|\cdot\|$ is a lattice norm, we have

$$\|\mu_+\| = \sup\{\mu(f) \mid f \in \mathcal{C}(K), f \geq 0 \text{ and } \|f\| \leq 1\}$$

where $\|\cdot\|$ also denotes the predual norm. Thus $\mu \mapsto \|\mu_+\|$ is w^* -lower semicontinuous, and likewise for $\mu \mapsto \|\mu_-\|$. Now, given general μ_k and ν_k satisfying

$$2\|\mu_k\|^2 + 2\|\nu_k\|^2 - \|\mu_k + \nu_k\|^2 \rightarrow 0$$

we get

$$2\|(\mu_k)_+\|^2 + 2\|(\nu_k)_+\|^2 - \|(\mu_k)_+ + (\nu_k)_+\|^2 \rightarrow 0$$

and similarly for $(\mu_k)_-$ and $(\nu_k)_-$. Therefore, we can apply the claim twice to get $(\mu_k - \nu_k)(H) \rightarrow 0$ for all $H \in \mathcal{H}$. \square

We apply the theorem above to obtain renorming results for more general Banach spaces. First, we give a modest generalisation of the classic transfer method for LUR renormings, applied to strictly convex renormings; cf. [1, Theorem II.2.1]. A proof is provided for completeness.

Proposition 2.7. *Let $(X, \|\cdot\|)^*$, $(Y, \|\cdot\|)^*$ be dual Banach spaces, with $(Y, \|\cdot\|)^*$ strictly convex. Further, let $|||\cdot|||$ be a coarser, w^* -lower semicontinuous seminorm on X^* , $T : X^* \rightarrow Y^*$ a bounded, linear operator and set $Z = \overline{T^*Y^*}^{|||\cdot|||} \subseteq X^*$. Then there exists an equivalent dual norm $|\cdot|$ on X , such that whenever*

$$f \in Z, f' \in X \quad \text{and} \quad |f| = |f'| = \tfrac{1}{2}|f + f'|$$

we have $|||f - f'| = 0$.

Proof. Define seminorms $|\cdot|_n$ on X^* by

$$|f|_n^2 = \inf\{|||f - T^*g||| + n^{-1}||g|| \mid g \in Y^*\}$$

and set $|f|^2 = ||f||^2 + \sum_{n \geq 1} 2^{-n}|f|_n^2$. Since $|||\cdot|||$ is coarser than $\|\cdot\|$, our new norm $|\cdot|$ is equivalent to $\|\cdot\|$. As in Theorem 2.6, by a w^* -compactness argument or the Hahn-Banach Theorem, $|\cdot|_n$ is a w^* -lower semicontinuous seminorm, and the infimum in the definition is attained. Now let f and f' satisfy the above hypothesis. By convexity arguments and infimum attainment, we can take $g_n, g'_n \in Y^*$ such that

$$(4) \quad |f|_n^2 = |||f - T^*g_n|||^2 + n^{-1}||g_n||^2,$$

$$(5) \quad |||f - T^*g_n||| = |||f' - T^*g'_n|||$$

and

$$||g_n|| = ||g'_n|| = \tfrac{1}{2}||g_n + g'_n||.$$

The last equation tells us that $g_n = g'_n$ for all n , meaning that we have

$$|||f - f'| \leq |||f - T^*g_n||| + |||f' - T^*g'_n|||$$

Since $f \in Z$, we have $|f|_n \rightarrow 0$, so by equations (4) and (5), this leads to $|||f' - T^*g'_n||| = |||f - T^*g_n||| \rightarrow 0$, giving $|||f - f'| = 0$ as required. \square

Using this, we can obtain our general renorming result.

Proposition 2.8. *Let $(X, \|\cdot\|)$ be a Banach space, $F \subseteq X^*$ a subspace and $|||\cdot|||$ a coarser norm on X , such that $F \cap (X, |||\cdot|||)^*$ separates points of X . Further, let $K \subseteq X$ be a Gruenhage compact in the $\sigma(X, F)$ -topology and suppose $X = \overline{\text{span}}^{|||\cdot|||}(K)$. Then:*

- (1) *there is a coarser, $\sigma(X, F)$ -lower semicontinuous, strictly convex norm $|\cdot|$ on X ;*
- (2) *X admits an equivalent, strictly convex norm.*

Moreover, if F is a norming subspace then $|\cdot|$ is equivalent to $\|\cdot\|$.

Proof. Since F is separating, we can identify $((X, \|\cdot\|), \sigma(X, F))$ as a topological subspace of $((F, \|\cdot\|)^*, w^*)$ by standard evaluation and consider K as a w^* -compact subset of F^* . Now elements of F act as continuous functions on (K, w^*) and the map $S : \mathcal{C}(K)^* \rightarrow F^*$, given by $(S\mu)(f) = \int_K f d\mu$, is a dual operator. Let $|||\cdot|||$ also denote the canonical norm on $G = (X, |||\cdot|||)^*$, and define the w^* -lower semicontinuous seminorm

$$|||\xi|||_1 = \sup\{\xi(f) \mid f \in F \text{ and } |||f||| \leq 1\}$$

on F^* . By Proposition 2.7, there exists an equivalent, dual norm $|\cdot|_1$ on F^* , such that if

$$\xi \in \overline{S\mathcal{C}(K)^*}^{|||\cdot|||_1}, \xi' \in F^* \quad \text{and} \quad |\xi|_1 = |\xi'|_1 = \frac{1}{2}|\xi + \xi'|_1$$

then $|||\xi - \xi'|_1 = 0$.

Let $|\cdot|$ be the restriction of $|\cdot|_1$ to X and note that $|\cdot|$ is both $\sigma(X, F)$ -lower semicontinuous and coarser than $|||\cdot|||$. Moreover, $X = \overline{\text{span}}^{|||\cdot|||}(K) \subseteq \overline{S\mathcal{C}(K)^*}^{|||\cdot|||_1}$. Therefore, whenever $|x| = |x'| = \frac{1}{2}|x + x'|$, we have $|||x - x'|_1 = 0$. Since $F \cap G$ separates points of X , it follows that $x' = x$. This gives (1). For (2), observe that the sum $|||\cdot||| + |\cdot|$ is an equivalent, strictly convex norm on X . Finally, if F is norming then $|\cdot|$ is equivalent to $|||\cdot|||$. \square

Let us assume that the coarser norm $|||\cdot|||$ of Proposition 2.8 is $\sigma(X, F)$ -lower semicontinuous. By a standard polar argument

$$|||x||| = \sup\{f(x) \mid f \in F, |||f||| \leq 1\}$$

and, in particular, $F \cap (X, |||\cdot|||)^*$ separates points of X .

Corollary 2.9. *Let X be a Banach space and $X^* = \overline{\text{span}}^{|||\cdot|||}(K)$, where K is a Gruenhage compact in the w^* -topology and $|||\cdot|||$ is equivalent to a coarser, w^* -lower semicontinuous norm on X^* . Then X^* admits an equivalent, strictly convex dual norm.*

The result above applies to all established classes of Banach spaces known to admit equivalent strictly convex dual norms on their dual spaces; for example, Vařák spaces. We move on to discuss a property of Banach spaces, introduced in [3] and shown there to be a sufficient condition for the existence of an equivalent, strictly convex dual norm.

Definition 2.10 ([3]). We say that the Banach space X has property G if there exists a bounded set $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n \subseteq X$, with the property that whenever $f, g \in B_{X^*}$ are distinct, there exist $n \in \mathbb{N}$ and $\gamma \in \Gamma_n$ such that $(f - g)(\gamma) \neq 0$ and, either $|f(\gamma')| > \frac{1}{4}|(f - g)(\gamma)|$ for finitely many $\gamma' \in \Gamma_n$, or $|g(\gamma')| > \frac{1}{4}|(f - g)(\gamma)|$ for finitely many $\gamma' \in \Gamma_n$.

As well as showing that all Vařák spaces possess property G, the authors of [3] remark that the property is closely related to Gruenhage compacta.

Proposition 2.11. *If X has property G then the dual unit ball B_{X^*} is a Gruenhage compact in the w^* -topology.*

Proof. We can and do assume that Γ is a subset of the unit ball B_X . Given $\gamma \in \Gamma$ and $q \in (0, 1) \cap \mathbb{Q}$, we let $U(\gamma, q) = \{f \in B_{X^*} \mid f(\gamma) > q\}$. We prove that, together, $(\mathcal{U}_{n,q})$ and $(\mathcal{V}_{n,q})$, $n \in \mathbb{N}$ and $q \in (0, 1) \cap \mathbb{Q}$, satisfy (1) and (2) of Definition 1.1, where $\mathcal{U}_{n,q} = \{U(\gamma, q) \mid \gamma \in \Gamma_n\}$ and $\mathcal{V}_{n,q} = \{-U(\gamma, q) \mid \gamma \in \Gamma_n\}$. Given distinct $f, g \in B_{X^*}$, take $\gamma \in \Gamma_n$ with the property that $\alpha = \frac{1}{4}|(f - g)(\gamma)| > 0$. It follows that either $|f(\gamma)| > \alpha$ or $|g(\gamma)| > \alpha$; without loss of generality, we assume that the former inequality holds. Now suppose that $f(\gamma) > 0$. We choose rational q to satisfy $f(\gamma) > q > \max\{g(\gamma), \alpha\}$ if $f(\gamma) > g(\gamma)$, or $g(\gamma) > q > f(\gamma)$ otherwise. Either way, $U(\gamma, q) \cap \{f, g\}$ is a singleton, giving (1). Since $q > \alpha$, (2) follows. If $f(\gamma) < 0$, we repeat the above argument with $-f$ and $-g$. \square

Corollary 2.12 ([3]). *If X has property G then X^* admits an equivalent, strictly convex dual norm.*

Proof. Combine Proposition 2.11 and Corollary 2.9. \square

We finish this section with an open problem.

Problem 2.13. *If $\mathcal{C}(K)^*$ admits a strictly convex dual norm then is K Gruenhage? More ambitiously, if X^* is a dual Banach space with strictly convex dual norm, is B_{X^*} Gruenhage?*

3. A TOPOLOGICAL CHARACTERISATION OF Y -EMBEDDABLE TREES

In this section, we present a partial converse to Theorem 2.6. We call a partially ordered set (Υ, \preceq) a *tree* if, for each $t \in \Upsilon$, the set $(0, t] = \{s \in \Upsilon \mid s \preceq t\}$ of *predecessors* of t is well-ordered. Given $t \in \Upsilon$, we denote by t^+ the set of *immediate successors* of t in Υ ; that is, $u \in t^+$ if and only if $t \prec u$ and $t \prec \xi \prec u$ for no ξ . The locally compact, scattered *order topology* on Υ takes as a basis the sets $(s, t]$, $s \prec t$, where $(s, t] = (0, t] \setminus (0, s]$. To ensure that this topology is also Hausdorff, we demand that every non-empty, totally ordered subset of Υ has at most one minimal upper bound; trees satisfying this property are themselves called *Hausdorff*. We study the space $\mathcal{C}_0(\Upsilon)$ of continuous, real-valued functions on Υ that vanish at infinity, and the dual space of measures. To date, most of the results about renorming $\mathcal{C}_0(\Upsilon)$ and its dual have been order-theoretic in character: [5], [13] and [14]. Such order-theoretic results, while well-suited in this context, are deeply bound to the tree-structure and, as such, do not offer obvious generalisations. Here, we are able to give a purely topological characterisation of trees Υ , such that $\mathcal{C}_0(\Upsilon)^*$ admits an equivalent, strictly convex dual norm. The following definition first appears in [13].

Definition 3.1. Let Y be the set of all strictly increasing, continuous, transfinite sequences $x = (x_\alpha)_{\alpha \leq \beta}$ of real numbers, where $0 \leq \beta < \omega_1$. We order Y by declaring that $x < y$ if and only if either y strictly extends x , or if there is some ordinal α such that $x_\xi = y_\xi$ for $\xi < \alpha$ and $y_\alpha < x_\alpha$.

We say that a map $\rho : \Upsilon \rightarrow \Sigma$ from a tree to a linear order is *increasing* if $\rho(s) \leq \rho(t)$ whenever $s \prec t$, and strictly so if the former inequality is always strict. The next theorem is the key result of this section.

Theorem 3.2. *If Υ is a tree and $\rho : \Upsilon \rightarrow Y$ is a strictly increasing function then Υ is a Gruenhage space.*

Theorem 2.6 and Theorem 3.2, together with [13, Proposition 7] and a result from [14], allows us to present the following series of equivalent conditions and, in particular, provides our partial converse to Theorem 2.6. Observe that a locally compact space is Gruenhage if and only if its 1-point compactification is.

Corollary 3.3. *If Υ is a tree then the following are equivalent:*

- (1) $\mathcal{C}_0(\Upsilon)^*$ admits an equivalent, dual p -UR norm;
- (2) $\mathcal{C}_0(\Upsilon)^*$ admits an equivalent, strictly convex dual lattice norm;
- (3) $\mathcal{C}_0(\Upsilon)$ admits an equivalent, Gâteaux smooth lattice norm;
- (4) $\mathcal{C}_0(\Upsilon)^*$ admits an equivalent, strictly convex dual norm;
- (5) there is a strictly increasing function $\rho : \Upsilon \rightarrow Y$;

(6) Υ is a Gruenhage space.

It is proved in [13] that the 1-point compactification of a tree Υ is descriptive, equivalently σ -discrete, if and only if there is a strictly increasing function $\rho : \Upsilon \rightarrow \mathbb{Q}$. As trees go, those that admit such \mathbb{Q} -valued functions are relatively simple. The order Y is considerably larger than \mathbb{Q} in order-theoretic terms; indeed, given any ordinal $\beta < \omega_1$, the lexicographic product \mathbb{R}^β embeds into Y . Accordingly, there is an abundance of trees that admit strictly increasing Y -valued maps, but not strictly increasing \mathbb{Q} -valued maps [13]. Therefore, the class of Gruenhage compact spaces encompasses appreciably more structure than the class of descriptive compact spaces.

A little preparatory work must be presented before giving the proof of Theorem 3.2. We recall some material from [13].

Definition 3.4 ([13]). A subset $V \subseteq \Upsilon$ is called a *plateau* if V has a least element 0_V and $V = \bigcup_{t \in V} [0_V, t]$. A partition \mathcal{P} of Υ consisting solely of plateaux is called a *plateau partition*.

If V is a plateau then $V \setminus \{0_V\}$ is open, so given a plateau partition \mathcal{P} of Υ , the set $H = \{0_V \mid V \in \mathcal{P}\}$ of *least elements* of V is closed in Υ .

Definition 3.5 ([13]). Given a tree Υ , let $(\mathcal{P}_\beta)_{\beta < \omega_1}$ be a sequence of plateau partitions with the following properties:

- (1) if $\alpha < \beta$ and $V \in \mathcal{P}_\alpha$, $W \in \mathcal{P}_\beta$, then either $W \subseteq V$ or $V \cap W$ is empty;
- (2) if β is a limit ordinal and $W \in \mathcal{P}_\beta$, then

$$W = \bigcap \{V \mid V \in \mathcal{P}_\alpha, \alpha < \beta, W \subseteq V\};$$

- (3) if $t \in \Upsilon$, there exists $\beta < \omega_1$, depending on t , such that $\{t\} \in \mathcal{P}_\beta$.

We call such a sequence of plateau partitions *admissible*.

Definition 3.6 ([13]). Let $(\mathcal{P}_\beta)_{\beta < \omega_1}$ be admissible and let T be the tree

$$\{(\alpha, V) \mid V \in \mathcal{P}_\alpha, \alpha < \omega_1\}$$

with order $(\alpha, V) \prec (\beta, W)$ if and only if $\alpha \leq \beta$ and $W \subseteq V$. Then the subtree

$$\Upsilon(\mathcal{P}) = \{(\beta, V) \in T \mid U \text{ is not a singleton whenever } (\alpha, U) \prec (\beta, V)\}$$

of T is called the *partition tree* of Υ with respect to $(\mathcal{P}_\beta)_{\beta < \omega_1}$.

It is evident that if V is a plateau then so is \overline{V} , with $0_{\overline{V}} = 0_V$. A subset of a tree Υ is called an *antichain* if it consists solely of pairwise incomparable elements. With respect to the interval topology, antichains are discrete subsets. We make the following elementary, yet important, observation.

Lemma 3.7. *Let E be an antichain in a partition tree $\Upsilon(\mathcal{P})$. If (α, V) and (β, W) are distinct elements of E then both intersections $V \cap W$ and $\overline{V} \setminus \{0_V\} \cap \overline{W} \setminus \{0_W\}$ are empty.*

Proof. We can assume that $\alpha \leq \beta$. That the first intersection is empty follows directly from the definition of the partition tree order. To see that the same is true for the second, note that if $(\alpha, U) \prec (\beta, W)$ then $\overline{W} \setminus \{0_W\} \subseteq \overline{U} \setminus \{0_U\}$, so all we need to do is prove that if $t \in \overline{V} \setminus \{0_V\} \cap \overline{U} \setminus \{0_U\}$ then V and U intersect non-trivially and are thus equal. Given such t , we have that 0_V and 0_U are comparable. If $0_V \preceq 0_U$ then since there exists $s \in (0_U, t] \cap V$, we have $0_U \in V$ as V is a plateau. Likewise, if $0_U \preceq 0_V$ then $0_V \in U$. \square

The next result shows that if there is a strictly increasing function $\rho : \Upsilon \longrightarrow Y$ then Υ admits a partition tree $\Upsilon(\mathcal{P})$, on which may be defined a strictly increasing, real-valued function. It is important to note that the order of the partition tree is related to the order of Υ through the second, albeit technical, property below. If $t \in \Upsilon$ then the *wedge* $[t, \infty)$ is the set $\{u \in \Upsilon \mid u \succcurlyeq t\}$.

Proposition 3.8 ([13]). *Let Υ be a tree. If $\rho : \Upsilon \longrightarrow Y$ is strictly increasing then there exists an admissible sequence of partitions $(\mathcal{P}_\beta)_{\beta < \omega_1}$ that yields a partition tree $\Upsilon(\mathcal{P})$, and a strictly increasing function $\pi : \Upsilon(\mathcal{P}) \longrightarrow [0, 1]$. Moreover:*

- (1) $\mathcal{P}_0 = \{[r, \infty) \mid r \in \Upsilon \text{ is minimal}\}$;
- (2) *for any non-maximal $(\beta, V) \in \Upsilon(\mathcal{P})$, the map*

$$0_W \longmapsto \pi(\beta + 1, W)$$

is strictly decreasing on the subtree of least elements

$$H_{(\beta, V)} = \{0_W \mid (\beta + 1, W) \in (\beta, V)^+\}.$$

In the proof below, we will assume the partition tree $\Upsilon(\mathcal{P})$ and function π from Proposition 3.8.

Proof of Theorem 3.2. We construct a legitimate system on Υ . As $\Upsilon(\mathcal{P})$ admits a strictly increasing, real-valued function π , its isolated elements may be decomposed into a countable union of antichains (F_n) . Indeed, if $(\beta, W) \in \Upsilon(\mathcal{P})$ is isolated and non-minimal, then it has an immediate predecessor (α, V) , and we can pick $\tau(\beta, W) \in \mathbb{Q} \cap (\pi(\alpha, V), \pi(\beta, W))$. Then consider the antichain of minimal elements, together with the fibres $(\tau^{-1}(q))_{q \in \mathbb{Q}}$. If V is a plateau then $\overline{V} \setminus V$ is an antichain and hence discrete. Note that here, closure is taken with respect to Υ . From Lemma 3.7, the family $\{\overline{V} \setminus \{0_V\} \mid (\beta, V) \in F_n\}$ is a pairwise disjoint collection of open sets in Υ . Hence $D_n = \bigcup \{\overline{V} \setminus V \mid (\beta, V) \in F_n\}$ is discrete.

Given $q \in \mathbb{Q}$, consider the set E_q of successor elements $(\beta + 1, W) \in (\beta, V)^+$, with $(\beta, V) \in \Upsilon(\mathcal{P})$ arbitrary, such that $\pi(\beta, V) < q < \pi(\beta + 1, W)$. Observe that E_q is an antichain in $\Upsilon(\mathcal{P})$. Indeed, if $(\alpha + 1, U) \prec (\beta + 1, W)$ and $(\beta + 1, W) \in E_q$ then $(\alpha + 1, U) \preccurlyeq (\beta, V) \prec (\beta + 1, W)$, thus $\pi(\alpha + 1, U) \leq \pi(\beta, V) < q$. It follows that $(\alpha + 1, U) \notin E_q$. Given non-maximal $(\beta, V) \in \Upsilon(\mathcal{P})$, property (2) of Proposition 3.8 tells us that, in particular, the set of relatively isolated points in the least elements $H_{(\beta, V)}$ can be decomposed into a countable union of antichains $(F_{(\beta, V), m})$ in Υ . Given $(\beta + 1, W) \in (\beta, V)^+$ such that $0_W \in F_{(\beta, V), m}$, set

$$E_{q, (\beta, V), W} = \{(\beta + 1, W') \in E_q \cap (\beta, V)^+ \mid 0_W \preccurlyeq 0_{W'}\}$$

and

$$\mathcal{E}_{q, m} = \{E_{q, (\beta, V), W} \mid (\beta + 1, W) \in (\beta, V)^+ \text{ and } 0_W \in F_{(\beta, V), m}\}.$$

We observe that each $\mathcal{E}_{q, m}$ is a family of disjoint subsets of E_q . Indeed, let $E_{q, (\beta, V), W}, E_{q, (\beta', V'), W'} \in \mathcal{E}_{q, m}$. If $(\beta, V) \neq (\beta', V')$ then $(\beta, V)^+ \cap (\beta', V')^+$ is empty and we are done, so we assume that this is not the case. If $W \neq W'$ then 0_W and $0_{W'}$ are incomparable in Υ , so $E_{q, (\beta, V), W}$ and $E_{q, (\beta', V'), W'}$ must be disjoint. By Lemma 3.7, it follows that the sets

$$J_{q, (\beta, V), W} = \bigcup \{W' \mid (\beta + 1, W') \in E_{q, (\beta, V), W}\},$$

$E_{q, (\beta, V), W} \in \mathcal{E}_{q, m}$, are also pairwise disjoint.

We prove that $J = J_{q, (\beta, V), W}$ is a plateau. Evidently 0_W is the least element of J . Now suppose $t \in J$ and $0_W \preccurlyeq s \preccurlyeq t$. We have to show that $s \in J$. As

$0_W, t \in V$ and V is a plateau, $s \in V$ and so there exists $(\beta + 1, W') \in (\beta, V)^+$ such that $s \in W'$. We know that $t \in W''$, where $(\beta + 1, W'') \in E_q \cap (\beta, V)^+$ and $0_W \preceq 0_{W'}''$. Thus we have $0_W \preceq 0_{W'} \preceq 0_{W''}$ and, by condition (2) of Proposition 3.8, $\pi(\beta + 1, W') \geq \pi(\beta + 1, W'') > q$. It follows that $(\beta + 1, W') \in E_q$ and $s \in J$.

At last, we have enough information to define our legitimate system. Begin by setting $A = \Upsilon$ and $\mathcal{H} = \{\{t\} \mid t \in \Upsilon \text{ is isolated}\}$. Then define $A_n = \overline{D_n}$ and $\mathcal{H}_n = \{\{t\} \mid t \in D_n\}$. Again using Lemma 3.7, we are permitted to define $A'_n = \Upsilon$ and $\mathcal{H}'_n = \{V \setminus \{0_V\} \mid (\beta, V) \in F_n\}$. From the above discussion, given $q \in \mathbb{Q}$ and $m \in \mathbb{N}$, we can define $A_{q,m} = \Upsilon$ and

$$\mathcal{H}_{q,m} = \{J_{q,(\beta,V),W} \setminus \{0_W\} \mid E_{q,(\beta,V),W} \in \mathcal{E}_{q,m}\}.$$

We claim that, together, the families $\mathcal{H}, \mathcal{H}_n, \mathcal{H}'_n$ and $\mathcal{H}_{q,m}$ separate points of Υ in the manner of Proposition 2.1, part (2). Let s, t be distinct elements of Υ . If s or t is an isolated point of Υ , we can separate using \mathcal{H} . Henceforth, we will assume that both s and t are limit elements of Υ . Let V_β^s be the unique element of β containing s , and likewise for t . Let $\gamma < \omega_1$ be minimal, subject to the condition that $V_\gamma^s \neq V_\gamma^t$. Such γ exists by property (3) of Definition 3.5. By property (2) of Definition 3.5, γ cannot be a limit ordinal. If $\gamma = 0$ then $V = V_\gamma^s = [r, \infty)$ by property (1) of Proposition 3.8. Being minimal in Υ , r is isolated, so $s \in V \setminus \{0_V\}$. As $(0, V)$ is minimal in $\Upsilon(\mathcal{P})$, it is an element of F_n for some n . Consequently, we can separate s from t using \mathcal{H}'_n .

We finish by tackling the case where $\gamma = \beta + 1$ for some ordinal β . Let $W = V_{\beta+1}^s$ and $W' = V_{\beta+1}^t$. If $s \in W \setminus \{0_W\}$ then as $(\beta + 1, W)$ is isolated in $\Upsilon(\mathcal{P})$, we can separate using some \mathcal{H}'_n as above. We can argue similarly if $t \in W' \setminus \{0_{W'}\}$ so, from now on, we assume that $s = 0_W$ and $t = 0_{W'}$, i.e. $s, t \in H_{(\beta,V)}$. If 0_W is an immediate successor with respect to $H_{(\beta,V)}$, i.e. if there exists $0_U \in H_{(\beta,V)}$ such that $0_U \prec 0_W$ and no element of $H_{(\beta,V)}$ lies strictly between the two, then $0_W \in \overline{U} \setminus U$. Indeed, if $r \prec 0_W$ then as 0_W is a limit in Υ , there exists $\xi \in (\max\{r, 0_U\}, 0_W] \setminus \{0_W\}$. Now ξ must lie in U because 0_U is the immediate predecessor of 0_W in $H_{(\beta,V)}$. It follows that $0_W \in \overline{U}$ as required. Now $(\beta + 1, U)$ is in F_n for some n , so $\{0_W\} \in \mathcal{H}_n$, thus separating 0_W from $0_{W'}$. As above, we can argue similarly if $0_{W'}$ is an immediate successor with respect to $H_{(\beta,V)}$, so now we assume that neither 0_W nor $0_{W'}$ are such elements. As $H_{(\beta,V)}$ has a least element and is a Hausdorff tree in its own right, the greatest element less than both 0_W and $0_{W'}$ is some $0_U \in H_{(\beta,V)}$ and, without loss of generality, we can assume that $0_U \prec 0_W$. If $0_{U'}$ is the immediate successor of 0_U in $H_{(\beta,V)}$ then $0_{U'} \prec 0_W$, because 0_W is not such an element. Consequently, $0_{U'} \in F_{(\beta,V),m}$ for some m so, given rational q strictly between $\pi(\beta, V)$ and $\pi(\beta + 1, W)$, we have $0_W \in J \setminus \{0_{U'}\}$, where $J = J_{q,(\beta,V),U'}$. Since $0_{U'} \not\preceq 0_{W'}$ by maximality of 0_U , it follows that $J \setminus \{0_{U'}\}$ separates 0_W from $0_{W'}$. \square

4. STABILITY PROPERTIES OF GRUENHAGE SPACES

Our first stability property is purely topological.

Theorem 4.1. *If X is a Gruenhage space and $f : X \longrightarrow Y$ is a perfect, surjective mapping, then Y is also Gruenhage.*

Proof. Let X be a Gruenhage space and assume that we have families (\mathcal{U}_n) and sets R_n satisfying Proposition 2.1 (3). By adding new families $\{\bigcup \mathcal{U}_n\}$ if necessary,

we assume that given n , there exists m such that $R_m = \bigcup \mathcal{U}_n$. If $G \subseteq \mathbb{N}$ is finite, define

$$\mathcal{V}_G = \{\bigcap_{i \in G} U_i \mid (U_i)_{i \in G} \in \prod_{i \in G} \mathcal{U}_i\}.$$

Given a perfect, surjective map $f : X \longrightarrow Y$, we set

$$\mathcal{V}_{F,G,k} = \{Y \setminus f(X \setminus (\bigcup_{i \in F} R_i \cup \bigcup \mathcal{F})) \mid \mathcal{F} \subseteq \mathcal{V}_G \text{ and } \text{card } \mathcal{F} = k\}$$

for finite $F, G \subseteq \mathbb{N}$ and $k \in \mathbb{N}$. Since f is perfect, every element of $\mathcal{V}_{F,G,k}$ is open in Y .

Let $y, z \in Y$ be distinct. We show that there exists finite $F, G \subseteq \mathbb{N}$, $k \in \mathbb{N}$ and $\mathcal{F} \subseteq \mathcal{V}_G$ with cardinality k , such that

$$\{y, z\} \cap Y \setminus f(X \setminus (\bigcup_{i \in F} R_i \cup \bigcup \mathcal{F}))$$

is a singleton. Moreover, if $\mathcal{G} \subseteq \mathcal{V}_G$ has cardinality k and

$$y \in Y \setminus f(X \setminus (\bigcup_{i \in F} R_i \cup \bigcup \mathcal{G}))$$

is non-empty, then $\mathcal{G} = \mathcal{F}$. From this, it follows immediately that Y is Gruenhage.

To prove this claim, we first construct a pair of decreasing sequences of compact sets. Set $A_0 = f^{-1}(y)$ and $B_0 = f^{-1}(z)$. Given $r \geq 0$, if, for all n , it is true that $(A_r \cup B_r) \cap R_n = \emptyset$ or $A_r \cap B_r \subseteq R_n$, then we stop. If not then let n_{r+1} be minimal, subject to the requirement that $(A_r \cup B_r) \cap R_{n_{r+1}} \neq \emptyset$ and $(A_r \cup B_r) \setminus R_{n_{r+1}} \neq \emptyset$. Put $A_{r+1} = A_r \setminus R_{n_{r+1}}$ and $B_{r+1} = B_r \setminus R_{n_{r+1}}$. Continuing in this way, either we stop at a finite stage or continue indefinitely.

If the process stops at a finite stage $r \geq 0$, set $A = A_r$ and $B = B_r$. Evidently $(A \cup B) \cap R_n = \emptyset$ or $A \cup B \subseteq R_n$ for all n . If the process above continues indefinitely, then we obtain a sequence $n_1 < n_2 < \dots$ and decreasing sequences (A_i) , (B_i) of non-empty, compact sets. Put $A = \bigcap_{i=0}^{\infty} A_i$ and $B = \bigcap_{i=0}^{\infty} B_i$. Then, given any n , again we have $(A \cup B) \cap R_n = \emptyset$ or $A \cup B \subseteq R_n$, lest we violate the minimality of the first $n_i > n$.

If $A = \emptyset$ then by surjectivity, and compactness if necessary, there is some $r \geq 1$ such that $A_r = \emptyset$. Since $(A_{r-1} \cup B_{r-1}) \setminus R_{n_r} \neq \emptyset$ by construction, it is not the case that B_r is empty, thus $f^{-1}(y) \subseteq \bigcup_{i=1}^r R_{n_i}$ and $f^{-1}(z) \not\subseteq \bigcup_{i=1}^r R_{n_i}$. Put $F = \{n_1, \dots, n_r\}$ and let G be arbitrary. Then $Y \setminus f(X \setminus \bigcup_{i \in F} R_i)$ is the only element of $\mathcal{V}_{F,G,0}$ and

$$\{y, z\} \cap Y \setminus f(X \setminus \bigcup_{i \in F} R_i) = \{y\}.$$

If $B = \emptyset$ then we proceed similarly.

Now suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Define $K = A \cup B$ and let

$$I = \{n \in \mathbb{N} \mid K \cap R_n = \emptyset \text{ and } K \subseteq \bigcup \mathcal{U}_n\}.$$

We have $K = \bigcup \{K \cap U \mid U \in \mathcal{U}_n\}$ whenever $n \in I$. Moreover, the sets in each $\{K \cap U \mid U \in \mathcal{U}_n\}$, $n \in I$, are pairwise disjoint. In fact slightly more can be said; if $x \in K \cap U \cap V$ for $U, V \in \mathcal{U}_n$ and $n \in I$ then $U = V$. Indeed, if $x \in K \cap U \cap V$, $U, V \in \mathcal{U}_n$ and $U \neq V$ then $x \in R_n$, so $n \notin I$.

Given distinct $a, b \in K$, there exists n and $U \in \mathcal{U}_n$ such that $\{a, b\} \cap U$ is a singleton. Firstly, this means $n \in I$. Indeed, if $K \cap R_n \neq \emptyset$ then $K \subseteq R_n$, meaning $a, b \in U$, which is not the case. Now suppose $K \not\subseteq \bigcup \mathcal{U}_n$. We have $\bigcup \mathcal{U}_n = R_m$ for some m , so $K \cap \bigcup \mathcal{U}_n = K \cap R_m$ is empty, which again is not the case. Thus $n \in I$. In particular, this means we can assume that $\{a, b\} \cap U = \{a\}$ because

$\{K \cap V \mid V \in \mathcal{U}_n\}$ partitions K . By compactness, it follows that there is finite $G \subseteq I$ and finite $\mathcal{E} \subseteq \bigcup_{i \in G} \mathcal{U}_i$ such that

$$A \subseteq \bigcup \{K \cap U \mid U \in \mathcal{E}\} \quad \text{and} \quad B \not\subseteq \bigcup \{K \cap U \mid U \in \mathcal{E}\}.$$

Let $x \in K \cap U$, where $U \in \mathcal{E}$. For every $i \in G$, we know from above that there is a unique $U_i \in \mathcal{U}_i$ such that $x \in U_i$. By definition $\bigcap_{i \in G} U_i \in \mathcal{V}_G$, and since $U \in \mathcal{U}_j$ for some $j \in G$, we have $U = U_j$ and $x \in \bigcap_{i \in G} U_i \subseteq U$. This allows us to take a finite subset $\mathcal{F} \subseteq \mathcal{V}_G$, such that

$$A \subseteq \bigcup \{K \cap V \mid V \in \mathcal{F}\} \quad \text{and} \quad B \not\subseteq \bigcup \{K \cap V \mid V \in \mathcal{F}\}.$$

We choose \mathcal{F} so that it has minimal cardinality k .

If necessary, we appeal to compactness to find $r \geq 0$ satisfying

$$f^{-1}(y) \subseteq \bigcup_{i=1}^r R_{n_i} \cup \bigcup \mathcal{F},$$

$A \subseteq f^{-1}(y) \setminus \bigcup_{i=1}^r R_{n_i}$ and $B \subseteq f^{-1}(z) \setminus \bigcup_{i=1}^r R_{n_i}$. Let $F = \{n_1, \dots, n_r\}$. Observe that if $\mathcal{G} \subseteq \mathcal{V}_G$ and $f^{-1}(y) \subseteq \bigcup_{i \in F} R_i \cup \bigcup \mathcal{G}$ then $A \subseteq \bigcup \mathcal{G}$, and likewise for $f^{-1}(z)$ and B . Thus $f^{-1}(z) \not\subseteq \bigcup_{i \in F} R_i \cup \bigcup \mathcal{F}$ and consequently

$$\{y, z\} \cap Y \setminus f(X \setminus (\bigcup_{i \in F} R_i \cup \bigcup \mathcal{F})) = \{y\}.$$

Now let $y \in Y \setminus f(X \setminus (\bigcup_{i \in F} R_i \cup \bigcup \mathcal{G}))$, where $\mathcal{G} \subseteq \mathcal{V}_G$ has cardinality k . It follows that $A \subseteq \bigcup \mathcal{G}$. We show that $\mathcal{G} = \mathcal{F}$. Take $W \in \mathcal{F}$. By minimality of k

$$A \not\subseteq \bigcup \{K \cap V \mid V \in \mathcal{F} \setminus \{W\}\}$$

thus there is $x \in A \cap W$. Take $V \in \mathcal{G}$ such that $x \in V$. We claim that $W = V$. Indeed, $W = \bigcap_{i \in G} W_i$ and $V = \bigcap_{i \in G} V_i$ for some $W_i, V_i \in \mathcal{U}_i$, $i \in G$. Since $G \subseteq I$ and $x \in K \cap W_i \cap V_i$, we have $W_i = V_i$ for all $i \in G$, hence $W = V \in \mathcal{G}$. Therefore $\mathcal{F} \subseteq \mathcal{G}$ and, by cardinality, we have equality as required. \square

Next, something of a more functional analytic nature.

Proposition 4.2. *If K is a Gruenhage compact then so is $B_{\mathcal{C}(K)^*}$.*

Proof. Let (A_n, \mathcal{H}_n) be a legitimate system satisfying properties (1) – (3), presented after Corollary 2.3. We can and do assume that $\emptyset \in \mathcal{H}_n$ for all n . Given $H \in \mathcal{H}_n$ and $q \in (0, 1) \cap \mathbb{Q}$, define the w^* -open set

$$U_+^{(H, n, q)} = \{\mu \in B_{\mathcal{C}(K)^*} \mid \mu_+(H \cup (K \setminus A_n)) > q\}$$

and let $\mathcal{U}_+^{(n, q)} = \{U_+^{(H, n, q)} \mid H \in \mathcal{H}_n\}$. Define $U_-^{(H, n, q)}$ and $\mathcal{U}_-^{(n, q)}$ in the corresponding manner. We claim that, with respect to $\mathcal{U}_+^{(n, q)}$ and $\mathcal{U}_-^{(n, q)}$, $n \in \mathbb{N}$ and $q \in (0, 1) \cap \mathbb{Q}$, $B_{\mathcal{C}(K)^*}$ is a Gruenhage compact in the sense of Definition 1.1.

Let $\mu, \nu \in B_{\mathcal{C}(K)^*}$ be distinct. Either $\mu_+ \neq \nu_+$ or $\mu_- \neq \nu_-$. We suppose that the former holds; if the latter holds then we repeat the argument below using the sets $U_-^{(H, n, q)}$ and $\mathcal{U}_-^{(n, q)}$. By Lemma 2.5, there exists $n \in \mathbb{N}$ and $H_0 \in \mathcal{H}_n$ such that $\mu_+(H_0) \neq \nu_+(H_0)$. If $\mu_+(K \setminus A_n) \neq \nu_+(K \setminus A_n)$ then set $H = \emptyset$. Otherwise, set $H = H_0$. Either way, we have $\mu_+(H \cup (K \setminus A_n)) \neq \nu_+(H \cup (K \setminus A_n))$ and, without loss of generality, we suppose that $\mu_+(H \cup (K \setminus A_n)) < q < \nu_+(H \cup (K \setminus A_n))$ for some rational q . Then $\{\mu, \nu\} \cap U_+^{(H, n, q)} = \{\nu\}$. Moreover, if $\mu \in U_+^{(H', n, q)}$ for some $H' \in \mathcal{H}_n$ then $\mu_+(H') = \mu_+(H' \cup (K \setminus A_n)) - \mu_+(K \setminus A_n) > q - \mu_+(H \cup (K \setminus A_n)) > 0$.

Hence, as each \mathcal{H}_n is a family of pairwise disjoint sets, μ can only be in finitely many elements of $\mathcal{U}_+^{(n,q)}$. \square

We finish by using these two results to glean a further crop of stability properties.

Proposition 4.3.

- (1) *If K is a Gruenhage compact and $\pi : K \longrightarrow M$ is continuous and surjective then M is also Gruenhage;*
- (2) *if $X_n, n \in \mathbb{N}$ are Gruenhage spaces then so is $\prod_n X_n$;*
- (3) *if X is a Banach space, $F \subseteq X^*$ is a separating subspace and $K \subseteq X$ is a Gruenhage compact in the $\sigma(X, F)$ -topology then so is its symmetric, $\sigma(X, F)$ -closed convex hull.*

Proof. (1) follows immediately from Theorem 4.1. To prove (2), we let X_n have a sequence $(\mathcal{U}_{n,m})_{m \in \mathbb{N}}$ of families of open sets satisfying Definition 1.1. It is straightforward to verify that the families $(\mathcal{V}_{n,m})$, defined by

$$\mathcal{V}_{n,m} = \{\prod_{i < n} X_i \times U \times \prod_{i > n} X_i \mid U \in \mathcal{U}_{n,m}\}$$

are witness to the fact that $\prod_{n \in \mathbb{N}} X_n$ is Gruenhage. To see that (3) holds, consider, as in Proposition 2.8, K as a subset of F^* and the map S restricted to $B_{\mathcal{C}(K)^*}$, which is Gruenhage by Proposition 4.2. By (1), $SB_{\mathcal{C}(K)^*} \subseteq F^*$ is a Gruenhage compact in the w^* -topology, giving (3). \square

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QUEENS' COLLEGE, CAMBRIDGE, CB3 9ET, UNITED KINGDOM
E-mail address: rjs209@cam.ac.uk